Lecture Notes 10: New Keynesian DSGE

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New Keynesian framework has emerged as the workhorse for the analysis of monetary policy and its implications for inflation, economic fluctuations, and welfare. It constitutes the backbone of new generation of medium-scale models under development at various central banks. The methodology underlying the New Keynesian models is similar to the RBC framework. The key features (frictions) that depart the former from the latter is the nominal rigidities and monopolistic competition. That is, the prices (wage rate, price of products) cannot be adjusted flexibly and thus the monetary policy is no longer natural. In this note, we will first introduce money into a standard RBC model via Cash-in-advance specification. We then relax the flexible-price assumption by introducing Calvo-type price setting problem. Finally, we will derive the New Keynesian Phillips curve, which plays a central role in monetary DSGE literature.

1 Cash-in-advance (CIA) Model

There is a final good, which can be either used for consumption or investment. The price of the final good is $P_t$. The timing for the representative household is following. At the beginning of the period $t$, the household has wealth $H_t$ which is carried from last period $(t - 1)$. And the monetary authority injects $X_t$ amount of money to the households. The total wealth available to the household at the beginning of the period $t$ is thus $H_t + X_t$. Then the household decides to hold $M_t$ amount of wealth in the form of cash, and $B_t$ amount of wealth in the form of risk-free bond. The constraint is given by

$$M_t + B_t \leq H_t + X_t.$$ 

(1)

The bond will pay interest rate $R_{bt}$ at the end of period $t$. The household then chooses consumption, which can be only purchased by the cash $M_t$. That is, the expenditure on consumption is subject to cash-in-advance (CIA) assumption:

$$P_t c_t \leq M_t.$$ 

(2)

Note that as holding cash does not earn interest rate, without the CIA constraint, the households have no incentive to hold cash, they would strictly prefer the bond. After the consumption decision, the households choose working hours $n_t$ with nominal wage rate $W_t$, and investments $k_{t+1} - (1 - \delta) k_t$. At the end of period, they receive labor income $W_t n_t$ and capital income $R_t k_t$. The wealth that the households carry to the next period $(t + 1)$ is defined as

$$H_{t+1} = M_t + (1 + R_{bt}) B_t + W_t n_t + R_t k_t - P_t [c_t + k_{t+1} - (1 - \delta) k_t].$$ 

(3)
The household’s problem is to maximize life-time utility

$$\max_{t=0}^{\infty} \beta^t \left( \log c_t - a_n n_t \right)$$

subject to (1), CIA constraint (2) and (3). To make things easy, we define real variables as:

- $$m_t = M_t/P_t$$,
- $$b_t = B_t/P_t$$,
- $$x_t = X_t/P_t$$,
- $$w_t = W_t/P_t$$,
- $$r_t = R_t/P_t$$,
- $$h_{t+1} = H_{t+1}/P_t$$.

Then the constraints (1), (2) and (3) can be reduced to

$$m_t + b_t \leq \frac{h_t}{1 + \pi_t} + x_t,$$

$$c_t \leq m_t,$$

$$h_{t+1} = m_t + (1 + R_{bt}) b_t + w_t n_t + r_t k_t - [c_t + k_{t+1} - (1 - \delta) k_t].$$

Denote the Lagrangian multiplies for the above constraints as $$\{\mu_t, \eta_t, \lambda_t\}$$ respectively. FOCs for $$\{m_t, b_t, h_{t+1}, c_t, n_t, k_{t+1}\}$$ are given by

$$\mu_t = \eta_t + \lambda_t,$$

$$\mu_t = \lambda_t (1 + R_{bt}),$$

$$\lambda_t = \beta E_t \left( \frac{1}{1 + \pi_{t+1} \mu_{t+1}} \right),$$

$$1/c_t = \lambda_t + \eta_t,$$

$$a_n = \lambda_t w_t,$$

$$\lambda_t = \beta E_t [\lambda_{t+1} (r_{t+1} + 1 - \delta)].$$

The firm side is standard, profit maximization implies that

$$r_t = \alpha y_t / k_t,$$

$$w_t = (1 - \alpha) y_t / n_t.$$

### 1.1 Competitive Equilibrium

The competitive equilibrium is defined as, each individual achieves the optimization and following markets clear.

**Good market:**

$$c_t + k_{t+1} - (1 - \delta) k_t = y_t,$$

**Bond market:**

$$b_t = b_{t-1} = 0.$$
Money market:

\[ mt = \frac{m_{t-1}}{1 + \pi_t} + x_t. \]  

(18)

Note that if two market clearing conditions satisfy, the remaining one automatically satisfies.

The full system consists of three constraints (5) to (7), eight FOCs (8) to (15), and two market clearing conditions (16) and (18). And we have 13 endogenous variables

\[ \{\mu_t, \eta_t, \lambda_t, m_t, b_t, h_{t+1}, c_t, n_t, k_{t+1}, r_t, w_t, \pi_t, R_b\}. \]

One important implication of the CIA model is that the model only gives the dynamics of real money balance \( m_t \) and inflation rate. Therefore, the dynamics is independent with the initial level nominal money stock \( M_0 \). An economy with higher money stock may have the same dynamics of the one with lower money stock. The only difference between two economies is the price level. Higher money stock implies higher price level. We call this property as neutrality of money.

### 1.2 Steady State

We first derive the steady state. Equations (9) and (10) implies the nominal interest rate \( R_b \) is given by

\[ R_b = \frac{1 + \pi}{\beta} - 1. \]  

(19)

The inflation rate \( \pi \) is determined by (18)

\[ \frac{\pi}{1 + \pi} = \frac{x}{m} = g, \]  

(20)

where \( g \) is the steady state growth rate of money supply. The Euler equation of capital decision (13) and capital demand (14) imply

\[ \frac{k}{y} = \frac{\alpha}{1/\beta - 1 + \delta}. \]  

(21)

With resource constraint (16), we can obtain \( \frac{c}{y} \). Given the steady state labor \( n \), we can obtain solve the \( \{k, y, c\} \). Moreover, (8), (9) and (11) imply

\[ \lambda = \frac{1}{c(1 + R_b)}. \]  

(22)

Then from (8), (9), \( \mu \) and \( \eta \) can be solved. Note that, as \( R_b > 0 \), (9) implies \( \mu > \lambda \). Furthermore, (8) implies \( \eta > 0 \). That is, the CIA constraint in the steady state is always binding as long as the interest rate is greater than zero. The economic intuition is that, because holding extra cash (which is larger than consumption) does not earn interest rate, therefore the household only hold the amount of cash that just meets the consumption need. It is easy to show that, the steady state values of other endogenous variables are easy to solve.
2 Basic New Keynesian Model

As we have already seen, monetary policy plays little role in the basic monetary model. We now turn to the New Keynesian (NK) model. The NK model has two basic features: (i) monopolistic competition; (ii) price is sticky. The former specification ensures that firms make positive profit, and the latter one implies that the price is not flexible and thus the money injection may play a non-trivial role.

2.1 Basic Economic Environment

The economy has one final good which can be used either for consumption or investment. The final good is produced by the final good firms. In particular, the final good firms use the intermediate goods as input. The final good market is competitive. The intermediate goods firms are monopolistic competitive, they have monopoly power to set their prices. However, the prices cannot be flexibly set. Therefore, the prices present some extent of stickness. Households are standard, just as those in the CIA model. We now start with the household problem.

2.2 Household Problem

The problem of representative household is similar to the CIA model. In particular, their optimization problem is defined as

$$\max \sum_{t=0}^{\infty} \beta^t (\log c_t - a_n n_t)$$

subject to

$$M_t + B_t \leq H_t + X_t,$$

$$P_t c_t \leq M_t,$$

$$H_{t+1} = M_t + (1 + R_{bt}) B_t + W_t n_t + R_k k_t + \Pi_t - P_t [c_t + k_{t+1} - (1 - \delta) k_t].$$

The only difference is that the profit of intermediate good firms, $\Pi_t$, appears in the above constraint. This is because monopolistic firms make positive profits.

2.3 Final Good Sector

Final good market is competitive, the firm combines a continuum intermediate goods $y_{it}$ as inputs to produce final good $y_t$. The production function is assumed to be

$$y_t = \left[ \int_0^1 y_{it}^{\frac{\sigma-1}{\sigma}} di \right]^{\frac{\sigma}{\sigma-1}}.$$
The profit maximization problem is
\[
\max_{y_{it}} P_t y_t - \int_0^1 P_{it} y_{it} di, \tag{28}
\]
subject to (27). The optimal \( y_{it} \) implies that the demand function of \( y_{it} \) is
\[
y_{it} = \left( \frac{P_{it}}{P_t} \right)^{-\sigma} y_t. \tag{29}
\]
Putting last equation into the production function, we get the price indexation function
\[
P_t^{-\sigma} = \int_0^1 P_{it}^{-\sigma} di. \tag{30}
\]
Note that as the final good market is competitive, with the CRS production function, the firm earns zero profit.

### 2.4 Intermediate Goods Sector

The intermediate goods sector is monopolistic. Firm \( i \) produces good \( i \) with Cobb-Douglas technology
\[
y_{it} = A_t k_{it}^{\phi} n_{it}^{1-\phi}. \tag{31}
\]
The real profit of firm \( i \) is defined as
\[
\pi_{it} = \frac{P_{it}}{P_t} y_{it} - w_t n_{it} - r_t k_{it}. \tag{32}
\]
where \( y_{it} = \left( \frac{P_{it}}{P_t} \right)^{-\sigma} y_t \). As the labor and capital decisions are static, the above profit function can be reduced to (through a cost-minimization problem)
\[
\pi_{it} = \left( \frac{P_{it}}{P_t} - \phi_t \right) \left( \frac{P_{it}}{P_t} \right)^{-\sigma} y_t, \tag{33}
\]
where \( \phi_t \) is the marginal cost
\[
\phi_t = \frac{1}{A_t} \left( \frac{w_t}{\alpha} \right)^\alpha \left( \frac{r_t}{1-\alpha} \right)^{1-\alpha}. \tag{34}
\]
The optimization problem for the firm \( i \) is to set price \( p_{it} \) to maximize the discounted profit flows. To model the price stickness, we follow Calvo (1982) assuming that in the period \( t \), the firm, with probability \( 1 - \theta \), can set its price flexibly. With probability \( \theta \), the firm cannot set price and thus the price remains the same as the previous period (\( t-1 \)). The Bellman equation for the firm that can adjust its price is
\[
V_{0,t} = \max_{p_{it}} \pi_{it} + \beta E_{t} \frac{\lambda_{t+1}}{\lambda_t} \left[ (1 - \theta) V_{0,t+1} + \theta V_{1,t+1} \right], \tag{35}
\]
where $V_{0,t}$ is the value of the firm that can adjust its price, and $V_{j,t}$ is the value of firm that adjust its price $j$ period ago and still cannot adjust its price in current period $(t)$. For instance, $V_{1,t+1}$ is the value of firm that adjust its price one period ago and still cannot adjust in period $t+1$. The optimal price $P_{i,t}$ for an active firm is set to satisfy

$$\frac{\partial \pi_{it}}{\partial P_{it}} + \beta \theta E_t \left[ \frac{\lambda_{t+1} \partial V_{1,t+1}}{\lambda_t} \right] = 0. \quad (36)$$

According to the profit function (33), the first term is

$$\frac{\partial \pi_{it}}{\partial P_{it}} = \frac{1}{P_t} \left( \frac{1}{P_{it}^\sigma} - \phi_t \right) \left( \frac{P_{it}^\sigma}{P_t} - \phi_t \right) \left( \frac{P_{it}^\sigma}{P_t} \right)^{-\sigma} y_{t}. \quad (37)$$

To derive the $\frac{\partial V_{1,t+1}}{\partial P_{it}}$, we need to specify the value function of inactive firm who adjust the price $j$ periods ago.

$$V_{j,t} (P_{it-j}) = \pi_{it} (P_{it-j}) + \beta E_t \frac{\lambda_{t+1}}{\lambda_t} \left[ (1 - \theta) V_{0,t+1} + \theta V_{j+1,t+1} \right]. \quad (38)$$

Note that there is no max operator in the above Bellman equation because the firm is inactive, it just takes the previous price as today’s price. Taking derivative w.r.t. $P_{it-j}$, we have

$$\frac{\partial V_{j,t}}{\partial P_{it-j}} = \frac{\partial \pi_{it} (P_{it-j})}{\partial P_{it-j}} + \beta \theta E_t \frac{\lambda_{t+1}}{\lambda_t} \frac{\partial V_{j+1,t+1}}{\partial P_{it-j}}. \quad (39)$$

From the above recursive structure of $\frac{\partial V_{j,t}}{\partial P_{it-j}}$, we can derive

$$\frac{\partial V_{j,t}}{\partial P_{it-j}} = \frac{\partial \pi_{it} (P_{it-j})}{\partial P_{it-j}} + \beta \theta E_t \frac{\lambda_{t+1}}{\lambda_t} \frac{\partial \pi_{it+1}}{\partial P_{it-j}} (P_{it-j}) + \beta \theta E_t \frac{\lambda_{t+1}}{\lambda_t} \frac{\partial \pi_{it+2}}{\partial P_{it-j}} (P_{it-j}) + ...$$

$$= \sum_{\tau=0}^{\infty} (\beta \theta)^\tau E_t \left[ \frac{\lambda_{t+\tau}}{\lambda_t} \frac{\partial \pi_{it+\tau}}{\partial P_{it-j}} (P_{it-j}) \right].$$

For the $\frac{\partial V_{1,t+1}}{\partial P_{it}}$, we then have

$$\frac{\partial V_{1,t+1}}{\partial P_{it}} = \sum_{\tau=0}^{\infty} (\beta \theta)^\tau E_t \left[ \frac{\lambda_{t+\tau}}{\lambda_t} \frac{\partial \pi_{it+\tau}}{\partial P_{it}} (P_{it}) \right]. \quad (40)$$

Plugging last equation into (36), we have

$$E_t \sum_{\tau=0}^{\infty} (\beta \theta)^\tau \left[ \frac{\lambda_{t+\tau}}{\lambda_t} \frac{\partial \pi_{it+\tau}}{\partial P_{it}} (P_{it}) \right] = 0 \quad (41)$$

where $\frac{\partial \pi_{it+\tau}}{\partial P_{it}} (P_{it}) = \frac{1}{P_{it+\tau}^\sigma} \left( \frac{1}{P_{it+\tau}^\sigma} - \phi_{t+\tau} \right) \left( \frac{P_{it}^\sigma}{P_{it+\tau}} - \phi_{t+\tau} \right) \left( \frac{P_{it}^\sigma}{P_{it+\tau}} \right)^{-\sigma} y_{t+\tau}$. With some algebra, we obtain the optimal pricing rule:

$$P_{it} = P_t^* = \frac{\sigma}{\sigma - 1} \frac{E_t \sum_{\tau=0}^{\infty} (\beta \theta)^\tau \lambda_{t+\tau} P_{it+\tau}^\sigma y_{t+\tau} \phi_{t+\tau}}{E_t \sum_{\tau=0}^{\infty} (\beta \theta)^\tau \lambda_{t+\tau} P_{it+\tau}^\sigma y_{t+\tau} \phi_{t+\tau}}. \quad (42)$$
Note that the optimal price $P_{it}$ is identical across firm index $i$, that is, once firms can adjust their price, they set the same optimal level. As a result, by law of large number, the price indexation function (30) implies

$$P_t^{-\sigma} = \theta P_{t-1}^{-\sigma} + (1 - \theta) (P_t^*)^{-\sigma}. \quad (43)$$

Now, we start to derive the so-called New Keynesian Phillips Curve (NKPC). First, in the steady state, (42) implies

$$P^* = \frac{\sigma}{\sigma - 1} \phi. \quad (44)$$

And (43) implies

$$P = P^*. \quad (45)$$

Log-linearizing (42) gives

$$\hat{P}_t^* = E_t (1 - \beta \theta) \sum_{\tau=0} (\beta \theta)^\tau \left( \hat{\lambda}_{t+\tau} + \sigma \hat{P}_{t+\tau} + \hat{y}_{t+\tau} + \hat{\phi}_{t+\tau} \right)$$

$$- E_t \sum_{\tau=0} (1 - \beta \theta) (\beta \theta)^\tau \left( \hat{\lambda}_{t+\tau} + (\sigma - 1) \hat{P}_{t+\tau} + \hat{y}_{t+\tau} \right)$$

$$= E_t \sum_{\tau=0} (1 - \beta \theta) (\beta \theta)^\tau \left( \hat{P}_{t+\tau} + \hat{\phi}_{t+\tau} \right)$$

$$= (1 - \beta \theta) \sum_{\tau=0} (\beta \theta L^{-1})^\tau \left( \hat{P}_t + \hat{\phi}_t \right)$$

$$= \frac{1 - \beta \theta}{1 - \beta \theta L^{-1}} \left( \hat{P}_t + \hat{\phi}_t \right) \quad (46)$$

or

$$\left( 1 - \beta \theta L^{-1} \right) \hat{P}_t^* = (1 - \beta \theta) \left( \hat{P}_t + \hat{\phi}_t \right), \quad (47)$$

where $L^{-1} (x_t) = E_t (x_{t+1})$. In addition, from (43), we have

$$\hat{P}_t = \theta \hat{P}_{t-1} + (1 - \theta) \hat{P}_t^*. \quad (48)$$

Combining last two equations, we have

$$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \frac{(1 - \theta) (1 - \beta \theta)}{\theta} \hat{\phi}_t, \quad (49)$$

where $\pi_t = \hat{P}_t - \hat{P}_{t-1}$ is the inflation rate.