Lecture Notes 2: Ramsey-Cass-Koopmans Model

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The Solow model does not consider individual optimal decisions. The model’s dynamic structure is simply introduced by the capital accumulation rule. The consumption rule is exogenously given, as a result, the saving rate is exogenous and constant. In this lecture, we will investigate the Ramsey-Cass-Koopmans model, in which the micro-level optimal behaviors are seriously modelled. In particular, the saving rate is instead endogenously determined by the household optimization decisions.

As the Ramsey model is in continuous time, before we formally discuss the details, we first introduce the standard solution method—Variation of Calculus.

1 Variation of Calculus

Consider a following integral:

$$J(t, x(t), \dot{x}(t)) = \int_{a}^{b} f(t, x(t), \dot{x}(t)) dt$$

where $a$ and $b$ are some constants. The function $x(t)$, captures the path of $x$ along the time $t$, can either be real-valued or $\mathbb{R}^n$ valued. For example, $x(t)$ can be the consumption path $c(t)$, or the vector of consumption path and saving path $[c(t); s(t)]$. Let $X$ be the set of all differentiable functions defined on the support $[a, b]$. A typical optimization problem in continuous time is defined as: choosing the optimal path $x(t) \in X$ to maximize (or minimize) $J$ subject to the terminal conditions:

$$x(a) = \alpha, \ x(b) = \beta.$$  \hspace{1cm} (2)

- Illustrative examples.

Example 1. Minimum Distance. Find the curve which joints two points on the plane with the minimum distance. A curve joining $A$ and $B$ can be represented by $x(t)$ with $x(a) = \alpha$, and $x(b) = \beta$.

The distance along each infinitesimal segment of $x(t)$ as $ds$, and we have

$$ds = \sqrt{(dt)^2 + (dx)^2} = \sqrt{1 + \left(\frac{dx}{dt}\right)^2} dt = \sqrt{1 + [\dot{x}(t)]^2} dt.$$  \hspace{1cm} (3)

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1This note is partly borrowed from the lecture notes of Prof. Yi Wen at Tsinghua University and Prof. Pengfei Wang at HKUST. I thank them for sharing their materials.
The total distance between points $\alpha$ and $\beta$ is then given by

$$J = \int_{a}^{b} \sqrt{1 + [\dot{x}(t)]^2} dt. \quad (4)$$

The problem is then equivalent to find a curve $x(t) \in X$ to minimize $J$ subject to $x(a) = \alpha$, and $x(b) = \beta$.

**Example 2. Saving Problem.** Suppose the overall utility on the interval $[0, T]$ is defined as

$$J = \int_{0}^{T} e^{-\rho t} u(c(t)) dt. \quad (5)$$

The budget constraint is given by

$$c(t) + \dot{s}(t) = rs(t). \quad (6)$$

Replacing $c(t)$ by $rs(t) - \dot{s}(t)$ gives us

$$J = \int_{0}^{T} e^{-\rho t} (rs(t) - \dot{s}(t)) dt. \quad (7)$$

The problem is to find a saving path $s(t)$ to maximize $J$. $lacksquare$

Now go back to the problem that finding an optimal $x(t)$ to optimize $J(t, x(t), \dot{x}(t)) = \int_{a}^{b} f(t, x(t), \dot{x}(t)) dt$ subject to the constraints $x(a) = \alpha$, $x(b) = \beta$. We assume that the function $f$ has continuous first and second partial derivatives with respect to all its arguments. Suppose that there exists a function $x^*(t) \in X$ that optimizes $J$.

Given any path $h(t)$ satisfying $h(a) = h(b) = 0$ and a small $\varepsilon > 0$, we define a new path $x_\varepsilon(t)$ such that

$$x_\varepsilon(t) = x^*(t) + \varepsilon h(t). \quad (8)$$

Therefore, the new path $x_\varepsilon(t)$ just deviates from $x^*(t)$ with a small perturbation. Define the value of $J$ on $x_\varepsilon(t)$ as $\bar{J}(\varepsilon)$, we then have

$$\bar{J}(\varepsilon) \equiv J(t, x_\varepsilon(t), \dot{x}_\varepsilon(t)) = \int_{a}^{b} f \left( t, x^*(t) + \varepsilon h(t), \dot{x}^*(t) + \varepsilon \dot{h}(t) \right) dt. \quad (9)$$

As $x^*(t) \in X$ optimizes $J$, we must have

$$\frac{\partial \bar{J}(\varepsilon)}{\partial \varepsilon} \bigg|_{\varepsilon=0} = \int_{a}^{b} \left[ f_{x(t)} h(t) + f_{\dot{x}(t)} \dot{h}(t) \right] dt = 0, \quad (10)$$
Note that, when $\varepsilon = 0$, the derivatives $f_x(t) = \frac{\partial f(t,x(t),\dot{x}(t))}{\partial x(t)}$, $\dot{f}_x(t) = \frac{\partial f(t,x(t),\dot{x}(t))}{\partial \dot{x}(t)}$ do not depend on $h(t)$. For the second term in the bracket, according to the integral by parts, we have

$$\int_a^b f_x(t) \dot{h}(t) \, dt = f_x(b)h(b) - f_x(a)h(a) - \int_a^b h(t) \left[ \frac{df_x(t)}{dt} \right] \, dt = -\int_a^b h(t) \left[ \frac{df_x(t)}{dt} \right] \, dt.$$  \hfill (11)

Putting last equation into (10) yields

$$\int_a^b \left[ f_x(t) - \frac{df_x(t)}{dt} \right] h(t) \, dt = 0. \hfill (12)$$

Since the function $h(t)$ takes any form with $h(a) = h(b) = 0$, last equation implies

$$f_x(t) = \frac{df_x(t)}{dt}, \text{ for } t \in [a, b]. \hfill (13)$$

This condition is called Euler equation which is the necessary condition for the optimal path $x^*(t)$ that optimizes $J$.

- **Minimum Distance Example** (Continued). In this case, since $f(t, x(t), \dot{x}(t)) = \sqrt{1 + [\dot{x}(t)]^2}$, we have

$$f_x(t) = 0, \quad f_{\dot{x}}(t) = \dot{x}(t) \left\{ 1 + [\dot{x}(t)]^2 \right\}^{-\frac{3}{2}}. \hfill (14)$$

The Euler equation (13) implies

$$\dot{x}^*(t) = \text{cons}, \hfill (15)$$

or

$$x^*(t) = \phi_1 t + \phi_2. \hfill (16)$$

To determine the $\phi_1$ and $\phi_2$, from terminal conditions $x(a) = \alpha$, $x(b) = \beta$, we have

$$\phi_1 = \frac{\beta - \alpha}{b - a}, \quad \phi_2 = \frac{ba - \beta a}{b - a}. \hfill (17)$$

Therefore, the optimal path $x^*(t) = \frac{\beta - \alpha}{b - a} t + \frac{ba - \beta a}{b - a}$. That is, the curve that minimizes the distance between point $\alpha$ and $\beta$ is a straight line. $\blacksquare$

2 Ramsey-Cass-Koopmans Model

Firms and households are representative with unit measure. The market is competitive. There is only one good, it can be used either for consumption or investment. The production technology is given by

$$Y_t = F(K_t, A_tL_t), \hfill (18)$$
where $L$ is the total labor input, $A_t$ is the exogenous technology, growing at rate $g > 0$,

$$\dot{A}_t = gA_t.$$  \hspace{1cm} (19)

We assume $F(.)$ is concave function and constant return to scale.

### 2.1 Firm’s Decision

Each period, from the market the representative firm hires labor $L_t$ with the wage rate $w_t$, and rents capital $K_t$ with the rental rate $r_t$. The firm chooses $L_t$ and $K_t$ to maximize its profit

$$\Pi_t = Y_t - r_tK_t - w_tL_t.$$ \hspace{1cm} (20)

The demand of capital and labor are given by following optimal conditions

$$r_t = \frac{\partial F(K_t, A_tL_t)}{\partial K_t},$$ \hspace{1cm} (21)

$$w_t = \frac{\partial F(K_t, A_tL_t)}{\partial L_t}.$$ \hspace{1cm} (22)

As $F(.)$ is constant return to scale, we have

$$\frac{\partial F(K_t, A_tL_t)}{\partial K_t}K_t + \frac{\partial F(K_t, A_tL_t)}{\partial L_t}L_t = Y_t.$$ \hspace{1cm} (23)

Plugging factor demands (21) and (22) into last equation gives us

$$Y_t = r_tK_t + w_tL_t,$$ \hspace{1cm} (24)

or $\Pi_t = 0$. That is, the constant return to scale of production function implies zero profit.

### 2.2 Household’s Decision

The representative household is a big family, in which there are $M_t$ members. Each individual inelastically supplies one unit labor. Therefore the total labor supply is $M_t$. The population $M_t$ is assumed to grow at the rate $n$

$$\dot{M}_t = nM_t.$$ \hspace{1cm} (25)

The representative household maximizes following life-time utility

$$U = \int_0^{\infty} e^{-\rho t}u(C_t)M_t dt,$$ \hspace{1cm} (26)
where \( C_t \) is the consumption of a member, \( u(C_t) \) is the corresponding utility level, and \( \rho \) is the discount rate for the future. In particular, the utility function is assumed to be

\[
u(C_t) = \frac{C_t^{1-\theta} - 1}{1-\theta},
\]

where \( \theta = -\frac{1}{u''} \) captures the coefficient of relative risk aversion. The utility function is concave if and only if \( \theta \geq 0 \).

The budget constraint for the household is given by

\[
C_t M_t + K_t = (r_t - \delta) K_t + w_t M_t.
\]

The optimization problem of representative household is to maximize (26) subject to (28).

### 2.3 Competitive Equilibrium

In the competitive equilibrium, the household and the firm achieve their individual optimum, and each market clears. In particular, \( M_t = L_t \). The budget constraint (28) and the input demands (21) and (22) jointly imply the resource constraint is

\[
C_t L_t + \dot{K}_t + \delta K_t = Y_t
\]

### 2.4 Dynamic System

Since there are two potential growth trends, to solve the model we need to first transform the economy to the stationary one. Define the detrended variables as \( x_t \equiv \frac{X_t}{A_t/L_t} \), while for the consumption of each individual, we define \( c_t = \frac{C_t}{A_t} \). The life-time utility can be rewritten as

\[
U = \int_0^\infty e^{-\rho t} \frac{c_t^{1-\theta}}{1-\theta} A_t^{1-\theta} L_t dt
\]

\[
= \int_0^\infty e^{-\rho t} \frac{c_t^{1-\theta}}{1-\theta} A_t^{1-\theta} L_t^n dt
\]

\[
= A_0^{1-\theta} L_0 \int_0^\infty e^{-\beta t} \frac{c_t^{1-\theta}}{1-\theta} dt
\]

\[\text{Consider a very short time interval } dt = \Delta, \text{ the budget constraint is}
\]

\[C_t M_t \Delta + K_{t+\Delta} - K_t = [(r_t - \delta) \Delta] K_t + w_t M_t \Delta.
\]

Dividing both sides by \( \Delta \) yields

\[
C_t M_t + \frac{dK_t}{dt} = (r_t - \delta) K_t + w_t M_t.
\]
where $\beta = \rho - (1 - \theta) g - n$. To guarantee that the utility function is not explosive, we need to assume $\beta > 0$. Without loss of generality, we set $A_0 = L_0 = 1$. Note that $\dot{K}_t = \frac{dK_t}{dt} = \frac{d\log K_t}{dt}$, where $\frac{d\log K_t}{dt} = \dot{k}_t + g + n$. The budget constraint can be rewritten as
\[
c_t + \dot{k}_t + (g + n + \delta) k_t = f(k_t),
\]
where $f(k) = F(\frac{K}{AL}, 1)$. Hence, the competitive equilibrium is equivalent to the solution of following social planner’s optimization problem
\[
\max_{\{c_t, k_t\}} \int_0^{\infty} e^{-\beta t} \frac{c_t^{1-\theta}}{1-\theta} \, dt
\]
subject to (30). Denote the Lagrangian multiplier of (30) as $\lambda_t$. The social planner’s problem can be written as
\[
\max_{\{c_t, k_t\}} \int_0^{\infty} e^{-\beta t} \left\{ \frac{c_t^{1-\theta}}{1-\theta} + \lambda_t \left[ f(k_t) - c_t - \dot{k}_t - (g + n + \delta) k_t \right] \right\} \, dt. \tag{32}
\]
Remember that the optimal $c_t$ and $k_t$ should satisfy the necessary condition (13). Therefore, we have
\[
c_t^{-\theta} = \lambda_t, \tag{33}
\]
\[
e^{-\beta t} \lambda_t \left[ f'(k_t) - (g + n + \delta) \right] = -\frac{d \left( e^{-\beta t} \lambda_t \right)}{dt} = -e^{-\beta t} \left( \dot{\lambda}_t - \beta \lambda_t \right). \tag{34}
\]
Combining last two equations, the optimal consumption path is given by
\[
\frac{\dot{c}_t}{c_t} = \frac{1}{\theta} \left[ f'(k_t) - (g + n + \delta + \beta) \right]. \tag{35}
\]
Moreover, the optimal capital $k_t$ satisfies resource constraint:
\[
\dot{k}_t = f(k_t) - c_t - (g + n + \delta) k_t. \tag{36}
\]
The dynamic system is fully described by the ordinary differential equation (OED) system (35) and (36). Note that given any initial state $(c_0, k_0)$, the system (35) and (36) only provide necessary conditions for the optimal path.

### 2.5 The Phase Diagram

According to (35), when $\dot{c}_t = 0$, the steady-state capital $k^*$ satisfies following equation
\[
f'(k) = \delta + g + n + \beta. \tag{37}
\]
\footnote{More rigorously, to guarantee $U$ is finite, we need the growth rate of $e^{-\beta t} \frac{c_t^{1-\theta}}{1-\theta}$ less than zero, or equivalently $(1 - \theta) \frac{d\log c_t}{dt} - \beta < 0$. As in the steady state, $\frac{d\log c_t}{dt} = 0$, we have $\beta > 0$.}
As \( f(k) \) is a concave function, \( f'(k) \) is decreasing in \( k \). If \( k_t > k^* \), we have \( f'(k_t) - (\delta + g + n + \beta) < 0 \), or \( \dot{c}_t < 0 \). If \( k_t < k^* \), we have \( \dot{c}_t > 0 \).

According to (36), when \( \dot{k}_t = 0 \), the steady-state \( c^* \) and \( k^* \) satisfy following equation

\[
c = f(k) - (g + n + \delta) k,
\]

which describes a hump-shaped curve in the \((c, k)\) space. For those \((c, k)\) above the curve, we have \( \dot{k}_t < 0 \). And for those \((c, k)\) below the curve, we have \( \dot{k}_t > 0 \).

Note that (37) is the set of all combinations of \((c, k)\) that ensure \( \dot{c}_t = 0 \), and the (38) is the set of all \((c, k)\) that ensure \( \dot{k}_t = 0 \). We call these two curves as the equilibrium locus. The intersection of them is the steady state.

The phase diagram consists of four areas separated by the locus (37) and the locus (38).

**Area I**: \( c_t \uparrow \) and \( k_t \downarrow \). Any \( c - k \) pair in this area indicates that capital stock is relatively high and consumption is relatively low, thus according to (35) and (36), consumption will increase and capital will decrease. The above dynamics imply that if the system starts from any point in this area, consumption and capital will eventually diverge from the steady-state point \((c^*, k^*)\).

**Area II**: \( c_t \downarrow \) and \( k_t \downarrow \). Any \( c - k \) pair in this area indicates that both capital stock and consumption are relatively high, thus according to (35) and (36), both of them will monotonically decrease as long as \((c_t, k_t)\) stays in this area. The above dynamics imply that in this area there exist some \( c - k \) pairs as initial points, from which consumption and capital will eventually converge to the steady-state point \((c^*, k^*)\).

**Area III**: \( c_t \uparrow \) and \( k_t \uparrow \). The dynamics in this area is just opposite to those in Area II.

**Area IV**: \( c_t \downarrow \) and \( k_t \uparrow \). The dynamics in this area is just opposite to those in Area I.

According to the above dynamic analysis, there exists a unique path (solid line with arrows in Figure 1) such that starting from any points in this path, the system will eventually converge to the steady state. We call this unique equilibrium path as the "saddle path".

Note that as the ordinary differential equations (35) and (36) just describes the dynamics of \( \dot{c}_t \) and \( \dot{k}_t \) instead of the levels of \( c_t \) and \( k_t \), these two equations only provide necessary conditions for the optimal path. To get the optimal path, we need to impose initial condition and terminal condition. Specifically, the stationarity condition (both \( c_t \) and \( k_t \) cannot diverge) gives the initial condition (will discuss later), and the non-ponzi-game condition,

\[
\lim_{s \to \infty} e^{-R_s} K_s = A_0 L_0 \lim_{s \to \infty} e^{-R_s} e^{(g+n)s} k_s \geq 0,
\]

defines the terminal condition. The non-ponzi-game condition means that one cannot constantly borrow from outside to cover it’s consumption. It can be shown that as long as the budget constraint
is satisfied, the non-ponzi-game condition always holds (see Romer’s textbook, page 53). Therefore, in the Ramsey model, we just need to consider the initial condition.

![Phase diagram of Ramsey model](image)

**Figure 1. Phase diagram of Ramsey model**

### 2.6 Modified Golden Rule and Balance Growth Path

Remember that the golden rule is defined as the steady-state (S-S) consumption at the maximum level. From (38), the capital stock at implied by the golden rule, $k^{GR}$, satisfies

$$f'(k^{GR}) = \delta + g + n.$$  \hspace{1cm} (40)

However, the optimal steady-state capital $k^*$ satisfies

$$f'(k^*) = \beta + \delta + g + n.$$  \hspace{1cm} (41)

Therefore, only if the discounting rate $\beta = 0$, the optimal S-S capital $k^*$ is identical to the golden rule capital $k^{GR}$. It is easy to see that if $\beta > 0$, $k^{GR} > k^*$. Moreover, the S-S saving rate in the
Ramsey model is
\[ s^* = 1 - \frac{c^*}{y^*} = 1 - \frac{f(k^*) - (g + n + \delta)k^*}{f(k^*)} = (g + n + \delta)k^* \frac{k^*}{f(k^*)} = \alpha \frac{g + n + \delta}{\beta + g + n + \delta} < \alpha. \] (42)

Last equality is due to \( \frac{f'(k^*)k^*}{f(k^*)} = \alpha \). The optimal S-S saving rate is less than the golden rule saving rate if \( \beta > 0 \). The intuition is that keeping the maximum consumption \( c^{GR} \) at each period is not optimal because the household cares more about current period than the future \( (\beta > 0) \). Therefore, starting at \( k = k^{GR} \) (higher than \( k^* \)), the household always has incentive to consume more than \( c^{GR} \) in the current period.

### 2.7 Transitional Dynamics: Algebraic Analysis

Now we provide a rigorous discussion about the optimal solution implied by (35) and (36). As the system is nonlinear, we need to first linearize the system around the steady state \( (c^*, k^*) \).

Define a new function \( \Psi(c_t, k_t) \) as
\[ \Psi(c_t, k_t) = \begin{bmatrix} \frac{1}{\beta} c_t \left[ f'(k_t) - (g + n + \delta + \beta) \right] & f(k_t) - c_t - (g + n + \delta) k_t \end{bmatrix}. \] (43)

Then the system (35) and (36) can be expressed as
\[ \begin{bmatrix} \dot{c}_t \\ \dot{k}_t \end{bmatrix} = \Psi(c_t, k_t). \] (44)

The first-order Taylor expansion around the steady state gives us
\[ \begin{bmatrix} \dot{c}_t \\ \dot{k}_t \end{bmatrix} \simeq \Psi (c^*, k^*) + \Psi_c (c^*, k^*) (c_t - c^*) + \Psi_k (c^*, k^*) (k_t - k^*). \] (45)

Note that \( \Psi (c^*, k^*) = 0 \), \( \Psi_c (c^*, k^*) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \); \( \Psi_k (c^*, k^*) = \begin{bmatrix} \frac{1}{\beta} c^* f''(k^*) \\ f'(k^*) - (g + n + \delta) \end{bmatrix} \). Define \( \tau = -\frac{1}{\beta} c^* f''(k^*) > 0 \) and joint with (37), we have \( f'(k^*) - (g + n + \delta) = \beta \). The linearized system then takes the form
\[ \begin{bmatrix} \dot{c}_t \\ \dot{k}_t \end{bmatrix} = \Gamma \begin{bmatrix} c_t - c^* \\ k_t - k^* \end{bmatrix}. \] (46)
where matrix $\Gamma = \begin{bmatrix} 0 & -\tau \\ -1 & \beta \end{bmatrix}$. From the above ordinary differential equations, we can derive the linearized equilibrium locus $\dot{c}_t = 0$, $\dot{k}_t = 0$. In particular, $\dot{c}_t = 0$ implies

$$k_t = k^*.$$  \hfill (47)

And $\dot{k}_t = 0$ implies

$$c_t - c^* = \beta (k_t - k^*).$$  \hfill (48)

The figure below depicts the phase diagram of the linearized system, which is quite similar to the phase diagram of the nonlinear system (Figure 1).

Now we show how to obtain the analytical solution of the saddle path. As $\Gamma$ is full-rank, it can be written as

$$\Gamma = P\Lambda P^{-1},$$  \hfill (49)

where $\Lambda$ is the eigenvalue matrix

$$\Lambda = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix},$$

$$\mu_1 = \frac{1}{2} \beta + \frac{1}{2} \sqrt{\beta^2 + 4\tau},$$

$$\mu_2 = \frac{1}{2} \beta - \frac{1}{2} \sqrt{\beta^2 + 4\tau}.$$  \hfill (50)
Obviously, \( \mu_1 > 0 \) and \( \mu_2 < 0 \). Furthermore, the eigenvector matrix \( P \) is
\[
P = \begin{bmatrix}
\frac{\mu_2}{\sqrt{1+\mu_2^2}} & \frac{\mu_1}{\sqrt{1+\mu_2^2}} \\
1 & \frac{1}{\sqrt{1+\mu_2^2}}
\end{bmatrix}.
\tag{51}
\]
The differential equations can be further expressed as
\[
\begin{bmatrix}
\dot{c}_t \\
k_t
\end{bmatrix} = P A P^{-1} \begin{bmatrix}
c_t - c^* \\
k_t - k^*
\end{bmatrix}.
\tag{52}
\]
It can be shown that the solution of the above system is given by
\[
\begin{bmatrix}
c_t - c^* \\
k_t - k^*
\end{bmatrix} = P \begin{bmatrix}
e^{\mu_1 t} & 0 \\
0 & e^{\mu_2 t}
\end{bmatrix} P^{-1} \begin{bmatrix}
c_0 - c^* \\
k_0 - k^*
\end{bmatrix},
\tag{53}
\]
or more compactly,
\[
c_t - c^* = \mu_2 a_1 e^{\mu_1 t} + \mu_1 a_2 e^{\mu_2 t},
\tag{54}
\]
\[
k_t - k^* = a_1 e^{\mu_1 t} + a_2 e^{\mu_2 t},
\tag{55}
\]
where \( a_1 \) and \( a_2 \) are functions of initial state \((c_0, k_0)\) and given by
\[
a_1 = \frac{1}{\mu_1 - \mu_2} [\mu_1 (k_0 - k^*) - (c_0 - c^*)],
\tag{56}
\]
\[
a_2 = -\frac{1}{\mu_1 - \mu_2} [\mu_2 (k_0 - k^*) - (c_0 - c^*)].
\tag{57}
\]
Note that (54) and (55) are just the solution derived from the necessary condition of the optimal decisions. Indeed, they are not the optimal solution (i.e., saddle path) of the problem (31). This is because given any pair of \((c_0, k_0)\), there exists a combination of paths \(\{c_t, k_t\}\) correspondingly. That is, there are potentially infinite combinations of paths \(\{c_t, k_t\}\) that solve (54) and (55). As a result, to find out the unique optimal path (saddle path), we need to impose extra conditions.

The first condition we have to impose is the stationarity condition. As you will see, this condition indeed is equivalent to the initial condition. Since we have \( \mu_1 > 0 \) and \( \mu_2 < 0 \), to make sure that \( c_t \) and \( k_t \) are not explosive, we have to set \( a_1 = 0 \). Then (56) implies
\[
c_0 - c^* = \mu_1 (k_0 - k^*).
\tag{58}
\]
Obviously, last equation is essentially the initial condition. Besides, in the Ramsey model, we do not need the terminal condition (non-ponzi-game condition), because the terminal condition is identical to the budget constraint.

Plugging (58) into (57) gives us
\[
a_2 = k_0 - k^*.
\tag{59}
From (54) and (55), together with $a_1 = 0$ and $a_2 = k_0 - k^*$, the saddle path is given by

$$k_t - k^* = (k_0 - k^*) e^{\mu_2 t},$$

$$c_t - c^* = (c_0 - c^*) e^{\mu_2 t}.$$  \hspace{1cm} (60)

In addition, last two equations and (58) imply

$$c_t - c^* = \mu_1 (k_t - k^*),$$

the slope of saddle path is $\mu_1$. As $\mu_1 > \beta$ (see equation 50), the saddle path is steeper than the locus $\dot{k}_t = 0$. This explains why in the phase diagram, the saddle path is in the middle of $\dot{c}_t = 0$ and $\dot{k}_t = 0$.

### 2.7.1 Transitional Dynamics: Unexpected Changes of $\beta$

So far, we have discussed how to obtain the saddle path, along which the economy will eventually converge to the steady state $(c^*, k^*)$. Given the fixed fundamental (no shocks, no changes of the values of deep parameters), the saddle path is a unique path that solves the optimization problem. Any initial $(c_0, k_0)$ off the saddle path will eventually diverge, and the corresponding path is not optimal to the economy.

In this section, we will discuss the scenario that if there is an unexpected change of the fundamental, what will the economy respond. To take a concrete example, we discuss the change of the discounting rate $\beta$.

Suppose that in the period 0, the economy is at the old steady state: $(c^*, k^*)$. In the same period, there is a sudden permanent increase in $\beta$. You may take the change of $\beta$ as a surprise or an exogenous shock. We denote the new $\beta$ as $\beta_{\text{new}}$, and the new steady state as $(c^{**}, k^{**})$. Under the new $\beta_{\text{new}}$, the dynamic system will be changed to

$$c_t - c^{**} = \mu_2 a_1^\text{new} e^{\mu_1^\text{new} t} + \mu_1 a_2^\text{new} e^{\mu_2^\text{new} t},$$

$$k_t - k^{**} = a_1^\text{new} e^{\mu_1^\text{new} t} + a_2^\text{new} e^{\mu_2^\text{new} t},$$

where

$$\mu_1^\text{new} = \frac{1}{2} \beta_{\text{new}} + \frac{1}{2} \sqrt{(\beta_{\text{new}})^2 + 4 \tau_{\text{new}}} > 0,$$  \hspace{1cm} (65)

$$\mu_2^\text{new} = \frac{1}{2} \beta_{\text{new}} - \frac{1}{2} \sqrt{(\beta_{\text{new}})^2 + 4 \tau_{\text{new}}} < 0.$$  \hspace{1cm} (66)

To see how the economy transits to the new steady state, we need to solve the coefficients $\{a_1^\text{new}, a_2^\text{new}\}$.

Note that according to (56) and (57), we have

$$a_1^\text{new} = \frac{1}{\mu_1^\text{new} - \mu_2^\text{new}} \left[ \mu_1^\text{new} (k^* - k^{**}) - (c^* + \Delta - c^{**}) \right],$$

$$a_2^\text{new} = -\frac{1}{\mu_1^\text{new} - \mu_2^\text{new}} \left[ \mu_2^\text{new} (k^* - k^{**}) - (c^* + \Delta - c^{**}) \right].$$

(67)

(68)
where $\Delta$ is the change of consumption (or impulse response) in period $t = 0$. The above two equations show that $\{a_{11}^{new}, a_{21}^{new}\}$ determine the change of consumption in the shock coming period, thus to get the transition path we need to solve these two coefficients.

Since $\mu_1^{new} > 0$, to make sure the path is not explosive, we have to set $a_{11}^{new} = 0$, or

$$\Delta = \mu_1^{new} (k^* - k^{**}) - (c^* - c^{**}).$$

The $\Delta$ describes the jump in consumption in the period 0 corresponding to the permanent change in $\beta$. Therefore, the transition path is fully described by

$$k_t - k^{**} = (k^* - k^{**}) e^{\mu_2^{new} x t},$$

$$c_t - c^{**} = (c^* + \Delta - c^{**}) e^{\mu_2^{new} x t}.$$  

Note that the $k_0 (= k^*)$ does not change in the period 0, because $k$ is a predetermined (or state) variable. Figure 3 illustrates the transition dynamics: $A \rightarrow B \rightarrow C$.

When the fundamental changes, the consumption adjusts such that $(c, k)$ will jump towards the new saddle path. Thus, we call the variable $c_t$ as control variable or jump variable.

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4In this case, the consumption immediately jumps to the saddle path, this is mainly due to the change of $\beta$ is unexpected and permanent.
2.7.2 Transitional Dynamics: Expected Changes of $\beta$

Suppose that in the period $0$, the economy is at the old steady state $(c^*, k^*)$. In the same period, there is an announcement saying that there will be a permanent increase in $\beta$ starting from $t = T$. That is, $\beta = \beta^{old}$ when $t < T$, and $\beta = \beta^{new}$, when $t \geq T$. We denote the new steady state as $(c^{**}, k^{**})$.

For those periods $t < T$, the dynamic system is still the old one:

$$c_t - c^* = \mu_2 a_1 e^{\mu_1 t} + \mu_1 a_2 e^{\mu_2 t}, \quad (72)$$

$$k_t - k^* = a_1 e^{\mu_1 t} + a_2 e^{\mu_2 t}. \quad (73)$$

Since we assume $k_0 = k^*$, from (56) and (57), we have

$$\mu_1 = \frac{1}{2} \beta + \frac{1}{2} \sqrt{\beta^2 + 4\tau}, \quad (74)$$

$$\mu_2 = \frac{1}{2} \beta - \frac{1}{2} \sqrt{\beta^2 + 4\tau}. \quad (75)$$

and

$$a_1 = -\frac{1}{\mu_1 - \mu_2} (c_0 - c^*), \quad (76)$$

$$a_2 = \frac{1}{\mu_1 - \mu_2} (c_0 - c^*). \quad (77)$$

For the periods $t \geq T$, the dynamic system under $\beta^{new}$ will be changed to

$$c_t - c^{**} = \mu_2^{new} a_1^{new} e^{\mu_1^{new} t} + \mu_1^{new} a_2^{new} e^{\mu_2^{new} t}, \quad (78)$$

$$k_t - k^{**} = a_1^{new} e^{\mu_1^{new} t} + a_2^{new} e^{\mu_2^{new} t}, \quad (79)$$

where

$$\mu_1^{new} = \frac{1}{2} \beta^{new} + \frac{1}{2} \sqrt{(\beta^{new})^2 + 4\tau^{new}} > 0, \quad (80)$$

$$\mu_2^{new} = \frac{1}{2} \beta^{new} - \frac{1}{2} \sqrt{(\beta^{new})^2 + 4\tau^{new}} < 0. \quad (80)$$

and

$$a_1^{new} = \frac{1}{\mu_1^{new} - \mu_2^{new}} [\mu_1^{new} (k_T - k^{**}) - (c_T - c^{**})], \quad (81)$$

$$a_2^{new} = -\frac{1}{\mu_1^{new} - \mu_2^{new}} [\mu_2^{new} (k_T - k^{**}) - (c_T - c^{**})]. \quad (82)$$

Since $\mu_1^{new} > 0$, to ensure that the path is not explosive, we have to set $a_1^{new} = 0$, or equivalently,

$$(c_T - c^{**}) = \mu_1^{new} (k_T - k^{**}). \quad (83)$$
Therefore, the transition path after $t \geq T$ is fully described by

\begin{align}
kt - k^{**} &= (k_T - k^{**}) e^{\mu_{new} x t}, \quad (84) \\
c_t - c^{**} &= (c_T - c^{**}) e^{\mu_{new} x t}. \quad (85)
\end{align}

To get the full transition path, we need to solve the part for $t < T$. From (72) and (73), you may find the transition path for $t < T$ is determined by the coefficients $\{a_1, a_2\}$, which are functions of $c_0$. To solve $c_0$, we first, according to (72) and (73), write $c_T$ and $k_T$ as functions of $c_0$:

\begin{align}
c_T &= \mu_2 a_1 e^{\mu_1 T} + \mu_1 a_2 e^{\mu_2 T} + c^*, \\
k_T &= a_1 e^{\mu_1 T} + a_2 e^{\mu_2 T} + k^*.
\end{align}

Putting last two equations into the initial condition (83) for the period $T$, we can finally solve $c_0$. Once we have $c_0$, the transition path during the period $t < T$, is described by (72) and (73).

The figure below illustrates the transition dynamics under an expected increase in $\beta$. It shows that even though the change of $\beta$ will occur in the future, the optimal transition for the economy is to jump in the initial period (remember that the $\beta$ is still unchanged for all $t < T$), and keep moving under the old system until the announcement is realized, at that time $(c_t, k_t)$ just arrives the new saddle path. The transition path in the Figure 4 is: $A \rightarrow B \rightarrow C$.

![Figure 4. Transition dynamics under expected change of $\beta$](image-url)