This note discusses households' consumption choice. In the next lecture, we will discuss firm's investment decision. It is safe to say that any propagation mechanism of macroeconomic model is closely related to the individuals' consumption and investment decisions. Thus, before we study the real business cycle theory, we first discuss the consumption and the investment through some stylized dynamic models.

1 Permanent-Income Hypothesis

We start with a simple dynamic consumption-saving model under certainty. Consider an individual who lives for $T$ periods whose optimization problem is

$$\max_{\{c_t,s_t\}} \sum_{t=1}^{T} u(c_t)$$

subject to budget constraints

$$c_t + s_t = y_t + s_{t-1}, \quad (1)$$

and terminal constraint

$$s_T \geq 0. \quad (2)$$

Note that the terminal constraint is indeed the "no-Ponzi-game" condition. The income flow $\{y_t\}$ is deterministic and exogenously given. As the period is finite, here we do not require a discounting rate to avoid divergent life-cycle utility.

To solve the model, we replace consumptions $\{c_t\}$ in the utility by $\{s_t\}$ and $\{y_t\}$. The optimization problem then can be expressed as

$$\max_{\{s_t\}} \sum_{t=1}^{T} u(y_t + s_{t-1} - s_t) = \max_{\{s_t\}} u(y_1 + s_0 - s_1) + u(y_2 + s_1 - s_2) + \ldots + u(y_{T-1} + s_{T-2} - s_{T-1}) + u(y_T + s_{T-1} - s_T) + \lambda s_T,$$

where $\lambda$ is the Lagrangian multiplier for the terminal condition (2). First order conditions w.r.t. $s_t$ are given by

$$u'(c_t) = u'(c_{t+1}), \quad (3)$$

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1 This note is partly borrowed from Prof. Yi Wen's lecture notes at Tsinghua University. I thank him for kindly sharing his materials.
for $t = 1, \ldots, T - 1$.

$$u'(c_T) = \lambda. \quad (4)$$

And the Kuhn-Tucker condition gives us

$$\lambda s_T = 0. \quad (5)$$

Note that this condition is analog to the transversality condition in the infinite horizon model. (3) implies that

$$c_t = c_{t-1}, \text{ for all } t. \quad (6)$$

As $u'(c) > 0$, (4) and (5) imply that $\lambda > 0$ and $s_T = 0$. Since the optimal consumption is constant over the time, from the budget constraint, we have

$$c_t = \frac{1}{T} s_0 + \frac{1}{T} \sum_{\tau=1}^{T} y_{\tau}, \text{ for all } t. \quad (7)$$

Intuitively, due to the concave utility function, the consumer wants to smooth his/her consumption flow. Since there is no uncertainty, the consumer can fully smooth his/her consumption for each period by consuming the average permanent income. The optimal consumption (7) has several important implications. First, it implies the consumption mainly depends on the average level of the life-time income, this is because the average income is much smoother than the income in particular period. Hence, transitory changes (or current change) in income do not have significant effects on the consumption. Second, saving plays the role of buffer-stock. That is, any extra income that is higher than the average income will be stored to compensate the periods with low income. To see this, we have

$$s_t - s_{t-1} = y_t - c_t = y_t - \left(\frac{1}{T} s_0 + \frac{1}{T} \sum_{\tau=1}^{T} y_{\tau}\right). \quad (8)$$

**Discussions**

1. When income is constant over time: $y_t = \bar{y}$, we have

$$c_t = \frac{1}{T} s_0 + \bar{y}, \quad (9)$$

$$s_t - s_{t-1} = -\frac{1}{T} s_0, \quad (10)$$

$$s_t = \left[1 - \frac{t}{T}\right] s_0. \quad (11)$$

In this case, income per-period reflects the permanent income and there is no temporary income, therefore, in each period, consumer uses all of the income to consume.
2. Change of income. Consider two cases:

a. Expected change of income. Suppose consumer expects that the income in period $\tau$ will increase an amount of $\Delta$, i.e., $\bar{y}_\tau = y_\tau + \Delta$, then the consumption is

$$c_t = \frac{1}{T} s_0 + \frac{1}{T} \sum_{j=1}^{T} y_j + \frac{1}{T} \Delta,$$  \hspace{1cm} \text{for all } t. \hspace{1cm} (12)

b. Unexpected change of income. Suppose in period $\tau$, the income suddenly increases $\Delta$, then the consumption path is

$$c_t = \begin{cases} 
\frac{1}{T} s_0 + \frac{1}{T} \sum_{j=1}^{\tau} y_j, & \text{for } t < \tau \\
\frac{1}{T} s_0 + \frac{1}{T} \sum_{j=1}^{T} y_j + \frac{1}{T-\tau+1} \Delta, & \text{for } t \geq \tau 
\end{cases}.$$ \hspace{1cm} (13)

3. Ricardian Equivalence. PIH says that the consumption only depends on the permanent income, thus any policy that does not change the income permanently would not affect the consumption. For example, if the government reduces tax today but will increase the tax in the future to balance the government spending, and if the consumers expect this, then the consumption will not change corresponding to the current tax reduction.

2 Consumption under Uncertainty

Suppose that the income $\{y_t\}$ follows a stochastic process, and the consumer’s optimization problem is give by

$$\max_{c_t} \mathbb{E}_0 \sum_{t=0}^{T} u(c_t)$$

subject to budget constraint:

$$c_t + s_t = y_t + s_{t-1}, \quad t = 0, 1, 2, ..., T,$$  \hspace{1cm} (14)

and terminal condition

$$s_T \geq 0.$$ \hspace{1cm} (15)

Putting the budget constraint into the utility function yields

$$\max_{c_t} \mathbb{E}_0 \left[ \sum_{t=0}^{T} u(y_t + s_{t-1} - s_t) + \lambda s_T \right].$$

The F.O.Cs are

$$u'(y_t + s_{t-1} - s_t) = \mathbb{E}_t u'(y_{t+1} + s_t - s_{t+1}),$$ \hspace{1cm} (16)
for $t = 0, \ldots, T - 1$.

$$u'(y_T + s_{T-1} - s_T) = \lambda,$$  \hspace{1cm} (17)

and

$$\lambda s_T = 0.$$ \hspace{1cm} (18)

Equation (16) implies

$$u'(c_t) = \mathbb{E}_t u'(c_{t+1}), \text{ for all } t,$$ \hspace{1cm} (19)

or

$$u'(c_t) = u'(c_{t-1}) + \varepsilon_t,$$ \hspace{1cm} (20)

where $\varepsilon_t$ is i.i.d. white noise satisfies $\mathbb{E}_{t-1}(\varepsilon_t) = 0$. The above process is called random walk.

Example: suppose the utility function takes the form

$$u(c) = c - \frac{\phi}{2} c^2.$$ \hspace{1cm} (21)

Then the optimal consumption satisfies

$$\mathbb{E}_t c_{t+1} = c_t,$$ \hspace{1cm} (22)

or equivalently

$$c_t = c_{t-1} + \varepsilon_t.$$ \hspace{1cm} (23)

To determine what $\varepsilon_t$ is, put last equation into budget constraint (14) and take expectation on both sides, we have

$$c_t = \frac{1}{T - t + 1} s_{t-1} + \frac{1}{T - t + 1} \mathbb{E}_t \sum_{\tau=0}^{T-t} y_{t+\tau},$$ \hspace{1cm} (24)

and

$$s_t = y_t + s_{t-1} - c_t$$

$$= y_t + \frac{T - t}{T - t + 1} s_{t-1} - \frac{1}{T - t + 1} \mathbb{E}_t \sum_{\tau=0}^{T-t} y_{t+\tau},$$ \hspace{1cm} (25)

for all $t$. Similarly, for $\{c_{t-1}, s_{t-1}\}$ we have

$$c_{t-1} = \frac{1}{T - t + 2} s_{t-2} + \frac{1}{T - t + 2} \mathbb{E}_{t-1} \sum_{\tau=0}^{T-t+1} y_{t-1+\tau},$$ \hspace{1cm} (26)

$$s_{t-1} = y_{t-1} + \frac{T - t + 1}{T - t + 2} s_{t-2} - \frac{1}{T - t + 2} \mathbb{E}_t \sum_{\tau=0}^{T-t+1} y_{t-1+\tau},$$ \hspace{1cm} (27)
Plugging last equation into (24), we get
\[
c_t = \frac{1}{T - t + 1} s_{t-1} + \frac{1}{T - t + 1} \mathbb{E}_t \sum_{\tau=0}^{T-t} y_{t+\tau}
\]
\[
= \frac{1}{T - t + 2} s_{t-2} - \frac{1}{T - t + 1} \left[ \frac{1}{T - t + 2} \mathbb{E}_t-1 \sum_{\tau=0}^{T-t+1} y_{t-1+\tau} - \mathbb{E}_t \sum_{\tau=0}^{T-t} y_{t+\tau} - y_{t-1} \right].
\] (28)

Finally, from (28) and (26), we can show that
\[
c_t - c_{t-1} = \frac{1}{T - t + 1} \left( \mathbb{E}_t \sum_{\tau=0}^{T-t} y_{t+\tau} - \mathbb{E}_{t-1} \sum_{\tau=0}^{T-t} y_{t+\tau} \right),
\] (29)

Last equation indicates that the \( \varepsilon_t \) is indeed the prediction error for the permanent income.

There is a very important implication in this example. Recall that the optimal consumption is given by (24), which implies that when making consumption choice, the consumer only cares about the expectation of the income flow. The risk or the volatility (second-order moment of \( y_t \)) does not affect consumer’s decision. We call this type of decision as "Certainty Equivalence". Note that, the certainty equivalence only appears when the marginal utility \( u'(c) \) is linear in \( c \), or \( u'' = 0 \). To see this, let us look at F.O.C (19). If \( u'(c) \) is linear, then we have
\[
u'(c_t) = \mathbb{E}_t \left[ u'(c_t) \right] = u'(\mathbb{E}_t[c_t]),
\] (30)
that is
\[c_t = \mathbb{E}_t [c_{t+1}], \text{ for all } t.
\] (31)

However, if \( u'(c) \) is not linear, or \( u''(c) > 0 \), then we have
\[
u'(c_t) = \mathbb{E}_t \left[ u'(c_{t+1}) \right] > u'(\mathbb{E}_t[c_{t+1}]),
\] (32)
or
\[c_t < \mathbb{E}_t [c_{t+1}].
\] (33)

In this case, the consumption is less than that in "Certainty Equivalence" case, or in other words, the saving in this case is relatively large. We call the "over-saving" behavior in this case as "precautionary saving".

### 2.1 Precautionary Saving in the Infinite Horizon Model

Let us consider an infinite horizon model. Consumer’s optimization problem is
\[
\max \mathbb{E}_0 \sum \beta^t \frac{c_t^{1-\theta} - 1}{1 - \theta}, \ \theta > 0,
\]
subject to budget constraint
\[ c_t + s_t = (1 + r) s_{t-1} + y_t. \]  

(34)

F.O.C is given by
\[ c_t^\theta = \beta (1 + r) E_t \left[ c_{t+1}^{\theta} \right]. \]  

(35)

For simplicity, we assume that the consumption \{c_t\} follows log-normal distribution, i.e.
\[ \log c_{t+1} \sim N \left( E_t \left( \log c_t \right), E_t \sigma_c^2 \right), \]  

(36)

then we have
\[ E_t \left( c_{t+1} \right) = e^{E_t \left( \log c_{t+1} \right) + \frac{1}{2} E_t \sigma_c^2}. \]  

(37)

Since \( c_t^{-\theta} = (e^{-\theta \log c_t}) \), we have
\[ E_t \left( c_{t+1}^{-\theta} \right) = e^{-\theta E_t \left( \log c_{t+1} \right) + \frac{\theta^2}{2} E_t \sigma_c^2}. \]  

(38)

Putting (38) into (35) and taking log on both sides, we have
\[ -\theta \log c_t = \log \left[ \beta (1 + r) \right] - \theta E_t \left( \log c_{t+1} \right) + \frac{\theta^2}{2} E_t \sigma_c^2, \]  

(39)

or
\[ E_t \left[ \frac{\Delta c_{t+1}}{c_t} \right] \simeq E_t \left[ \frac{\Delta c_{t+1}}{c_t} \right] = \frac{1}{\theta} \log \left[ \beta (1 + r) \right] + \frac{\theta}{2} E_t \sigma_c^2. \]  

(40)

Precautionary saving implies that the expected consumption growth is affected by the uncertainty.

### 2.2 Precautionary Saving under Credit Constraint

Indeed, the precautionary saving behavior does not necessarily require convex marginal utility, as long as the economy suffers credit constraint. To see this, let us consider a simple three-period model. The consumer’s optimization problem is
\[ \max E_1 \left[ u \left( c_1 \right) + u \left( c_2 \right) + u \left( c_3 \right) \right] \]

subject to budget constraints
\[
\begin{align*}
c_1 + s_1 &= y_1 + s_0, \\
c_2 + s_2 &= y_2 + s_1, \\
c_3 + s_3 &= y_3 + s_2,
\end{align*}
\]

and terminal condition
\[ s_3 \geq 0, \]  

(41)
Besides, we introduce credit constraint for period 2

\[ s_2 \geq 0. \] (42)

As we already know that without credit constraint, the optimal consumption in period 1 is

\[ c_1 = \frac{1}{3} s_0 + \frac{1}{3} E_1 (y_1 + y_2 + y_3). \] (43)

Now if we consider the credit constraint, then the problem becomes

\[
\max E_1 \left[ \begin{array}{c}
  u(c_1) + u(c_2) + u(c_3) \\
  + \lambda_1 (y_1 + s_0 - s_1 - c_1) \\
  + \lambda_2 (y_2 + s_1 - s_2 - c_2) \\
  + \lambda_3 (y_3 + s_2 - s_3 - c_3) \\
  + \pi_1 s_2 + \pi_2 s_3
\end{array} \right].
\]

FOCs are

\[
\begin{align*}
  u'(c_1) &= \lambda_1, \\
  u'(c_2) &= \lambda_2, \\
  u'(c_3) &= \lambda_3, \\
  \lambda_1 &= E_1 \lambda_2, \\
  \lambda_2 &= \pi_1 + E_2 \lambda_3, \\
  \lambda_3 &= \pi_2.
\end{align*}
\]

It can be shown that

\[
  u'(c_1) = E_1 \pi_1 + E_1 u'(c_3). \] (44)

Since the Lagragian multiplier \( \pi_1 \geq 0 \), we have \( u'(c_1) = E_1 u'(c_2) \geq E_1 u'(c_3) \). Suppose \( u'(c) \) is linear, then we have

\[
\begin{align*}
  c_1 &= E_1 (c_2), \\
  c_1 &\leq E_1 (c_3).
\end{align*}
\]

Furthermore, from the budget constraint, we have

\[
c_1 + E_1 c_2 + E_1 c_3 = s_0 + E_1 (y_1 + y_2 + y_3),
\] (45)

or

\[
c_1 \leq \frac{1}{3} s_0 + \frac{1}{3} E_1 (y_1 + y_2 + y_3). \] (46)
3 Consumption and Risky Assets

So far, we only consider the risk-free asset, that is, the interest rate \( r_{t+1} \) is deterministic in period \( t \). We now consider a bunch of assets (indexed by \( i \)) with stochastic rates of return. F.O.C for asset \( i \) is given by

\[
1 = \beta \mathbb{E}_t \left[ (1 + r^i_{t+1}) (1 + g^c_{t+1})^{-\theta} \right].
\]

For the term \( (1 + r^i_{t+1}) (1 + g^c_{t+1})^{-\theta} \), after second-order Taylor expansion around \( r^i_{t+1} = g^c_{t+1} = 0 \), we have

\[
(1 + r^i_{t+1}) (1 + g^c_{t+1})^{-\theta} = 1 + r^i_{t+1} - \theta g^c_{t+1} - \theta r^i_{t+1} g^c_{t+1} + \frac{\theta (\theta + 1)}{2} (g^c_{t+1})^2.
\]

And thus for any \( i \), we have

\[
\frac{1}{\beta} - 1 = \mathbb{E}_t \left( r^i_{t+1} \right) - \theta \mathbb{E}_t \left( g^c_{t+1} \right) - \theta \mathbb{E}_t \left( r^i_{t+1} \right) \mathbb{E}_t \left( g^c_{t+1} \right) - \theta \text{cov}_t \left( r^i_{t+1}, g^c_{t+1} \right)
+ \frac{\theta (\theta + 1)}{2} \left[ \mathbb{E}_t \left( g^c_{t+1} \right) \right]^2 + \frac{\theta (\theta + 1)}{2} \text{var}_t \left( g^c_{t+1} \right).
\]

If we ignore terms \( \mathbb{E}_t \left( r^i_{t+1} \right) \mathbb{E}_t \left( g^c_{t+1} \right) \) and \( \left[ \mathbb{E}_t \left( g^c_{t+1} \right) \right]^2 \) which are very small, the difference of rates of return between asset \( i \) and \( j \) is

\[
\mathbb{E}_t \left( r^i_{t+1} \right) - \mathbb{E}_t \left( r^j_{t+1} \right) = \theta \left[ \text{cov}_t \left( r^i_{t+1}, g^c_{t+1} \right) - \text{cov}_t \left( r^j_{t+1}, g^c_{t+1} \right) \right]
= \theta \text{cov}_t \left( r^i_{t+1} - r^j_{t+1}, g^c_{t+1} \right)
= \theta \text{corr} \left( r^i_{t+1} - r^j_{t+1}, g^c_{t+1} \right) \text{std} \left( r^i_{t+1} - r^j_{t+1} \right) \text{std} \left( g^c_{t+1} \right).
\]

If we take \( j \) as riskless asset, then the LHS is the risk premium. In the U.S. data (1979-2003), the risk premium is about 7\%, and the correlation between \( r^i_{t+1} - r^j_{t+1} \) and \( g^c_{t+1} \) is about 0.27, the standard deviation of excess return is 14.4\%, and the standard deviation of consumption growth rate is 1.1\%. Therefore, the implied coefficient of relative risk aversion is

\[
\theta = \frac{0.07}{0.27 \times 0.01 \times 0.144} = 163.
\]

In order to explain the risk premium observed in the data, the coefficient \( \theta \) must be unreasonably large, this is so-called risk premium puzzle.